A Moore Bound for Simplicial Complexes

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Abstract

Let X be a d-dimensional simplicial complex with N faces of dimension $(d-1)$. Suppose that any $(d-1)$-face of X is contained in at least $k \geq d+2$ faces of X of dimension d. Extending the classical Moore bound for graphs, it is shown that X must contain a ball B of radius at most $\lceil \log_{k-d} N \rceil$ such that $H_d(B;\mathbb{R}) \neq 0$. The Ramanujan Complexes constructed by Lubotzky, Samuels and Vishne are used to show that this upper bound on the radius of B cannot be improved by more then a multiplicative constant factor.

1 Introduction

Let $G = (V, E)$ be a graph on n vertices. Let $\delta(G)$ denote the minimal degree in G and let $g(G) = g$ denote the minimal length of a cycle in G. An easy counting argument (see e.g. Theorem IV.1 in [2]) shows that if $\delta(G) = k \geq 3$ then

$$n \geq \begin{cases} 1 + \frac{k}{k-2}((k-1)^{\frac{g-1}{2}} - 1) & g \text{ odd} \\ \frac{2}{k-2}((k-1)^{\frac{g}{2}} - 1) & g \text{ even} \end{cases} . \tag{1}$$

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This implies the classical Moore bound

Theorem A. $g(G) < 2\log_{k-1} n + 2$.

Let $d_G(u,v)$ be the distance between the vertices u and v in the graph metric and let $B_r(v) = \{u \in V : d_G(u,v) \leq r\}$ denote the ball of radius r around v. Define the *acyclicity radius* $r_v(G)$ of G at the vertex v to be the maximal r such that the induced graph $G[B_r(v)]$ is acyclic. Let $r(G) = \min_{v\in V} r_v(G)$, then $r(G) = \lfloor \frac{g(G)}{2} \rfloor - 1$. The asymptotic version of Moore's bound is equivalent to the following

Theorem A$_1$. *If $\delta(G) = k \geq 3$ then for every $v \in V$*

$$r_v(G) \leq \lfloor \log_{k-1} n \rfloor \quad . \tag{2}$$

The best lower bound for the girth of k-regular graphs is given by the Ramanujan graphs of Lubotzky, Phillips and Sarnak [6]. For a fixed prime p, the construction in [6] provides a sequence of $(p+1)$-regular graphs $G_i = (V_i, E_i)$ with $|V_i| \to \infty$ such that $g(G_i) \geq \frac{4}{3}\log_p |V_i| - O(1)$. A similar result was obtained by Morgenstern [9] for any prime power q. In terms of the acyclicity radius we therefore have:

Theorem B$_1$. *For every prime power q, there exists a sequence of $(q+1)$-regular graphs $G_i = (V_i, E_i)$ with $|V_i| \to \infty$ such that for every $v \in V$*

$$r_v(G_i) \geq \frac{2}{3}\log_q |V_i| - O(1) \quad .$$

In this note we extend Theorems A$_1$ and B$_1$ to higher dimensional simplicial complexes. Let X be a d-dimensional simplicial complex on the vertex set V. Let $H_i(X)$ denote the i-dimensional homology group of X with some fixed field coefficients. For $0 \leq i \leq d$ let $X(i) = \{\sigma \in X : \dim \sigma = i\}$ and let $f_i(X) = |X(i)|$. For a subset of vertices $S \subset V$ let $X[S]$ denote the induced subcomplex on S. The *degree* of a $(d-1)$-simplex $\sigma \in X(d-1)$ is

$$\deg(\sigma) = |\{\tau \in X(d) : \sigma \subset \tau\}| \quad .$$

Let $\delta(X) = \min\{\deg(\sigma) : \sigma \in X(d-1)\}$. A complex X is called *k-regular* if $\deg(\sigma) = \delta(X) = k$ for every $\sigma \in X(d-1)$. Denote by $B_r(v)$ the ball of radius r around v with respect to the graph metric on the 1-dimensional skeleton of

X. Extending the notion of acyclicity radius to the higher dimensional setting we define $r_v(X)$ as the maximal r such that $H_d(X[B_r(v)]) = 0$, and $r(X) = \min_{v\in V} r_v(X)$. The following result extends Theorem A_1 to d-dimensional complexes.

Theorem $\mathbf{A}_d$. *Let X be a d-dimensional complex with $\delta(X) = k \geq d+2$. Then for any vertex $v \in V$ which is contained in some $(d-1)$-face*

$$r_v(X) \leq \lfloor \log_{k-d} f_{d-1}(X) \rfloor \quad .$$

For the lower bound, we use the Ramanujan Complexes presented by Lubotzky, Samuels and Vishne in [8] to show:

Theorem $\mathbf{B}_d$. *For $d \geq 1$ and q a prime power, there exists a sequence of d-dimensional $(q+1)$-regular complexes X_i on vertex sets V_i with $|V_i| \to \infty$, such that for any $v \in V$*

$$r_v(X_i) \geq \frac{\log_q |V_i|}{2d^2(d+2)} - 1 \quad .$$

Theorem A_d is proved in Section 2, while Theorem B_d is established in Section 3. Note that Theorem A_d reduces to Theorem A_1 when $d = 1$. On the other hand, specializing theorem B_d for the case $d = 1$, yields a somewhat weaker version of Theorem B_1 (The constant is $\frac{1}{6}$ rather then $\frac{2}{3}$). In Section 4 we discuss some open problems and suggestions for further research. One such challenge is to improve the constant in Theorem B_d.

2 The Upper Bound

Proof of Theorem $\mathbf{A}_d$: First note that if Y is a d-dimensional complex such that $f_d(Y) > f_{d-1}(Y)$, then $H_d(Y) \neq 0$. Indeed, let $C_i(Y)$ denote the space of simplicial i-chains of Y. Then $\dim C_d(Y) = f_d(Y) > f_{d-1}(Y) = \dim C_{d-1}(Y)$ implies that the boundary map $\partial : C_d(Y) \to C_{d-1}(Y)$ has a non-trivial kernel.

Let v be a vertex which is contained in a $(d-1)$-simplex. Abbreviate $B_t = B_t(v)$ and write $\alpha(t) = f_{d-1}(X[B_t])$, $\beta(t) = f_d(X[B_t])$. Let

$$\gamma(t) = |\{(\sigma,\tau) \ : \ \sigma \in X[B_t](d-1) \ , \ \tau \in X(d) \ , \ \sigma \subset \tau\}| \quad .$$

Then

$$\gamma(t) = \sum_{\sigma \in X[B_t](d-1)} \deg(\sigma) \geq f_{d-1}(X[B_t]) \cdot \delta(X) \geq \alpha(t) \cdot k \ . \tag{3}$$

For a d-simplex $\tau \in X(d)$ let $s(\tau)$ denote the number of $(d-1)$-simplices in $X[B_t]$ that are contained in τ. Then

$$s(\tau) = \begin{cases} d+1 & \tau \in X[B_t] \\ 0 & \tau \notin X[B_{t+1}] \end{cases}$$

and $s(\tau) \leq 1$ if $\tau \in X[B_{t+1}] - X[B_t]$. Thus

$$\gamma(t) = \sum_{\tau \in X(d)} s(\tau) \leq (d+1)\beta(t) + (\beta(t+1) - \beta(t)) =$$

$$d\beta(t) + \beta(t+1) \ \ . \tag{4}$$

Let $m = r_v(X)$. Combining (3) and (4) we obtain that for all $t < m$

$$k\alpha(t) \leq d\beta(t) + \beta(t+1) \leq d\alpha(t) + \alpha(t+1) \ .$$

Hence

$$\alpha(t+1) \geq (k-d)\alpha(t) \geq \cdots \geq (k-d)^t \alpha(1) \ .$$

Since v is contained in a $(d-1)$-face, it follows that $\alpha(1) \geq kd + 1$. Thus

$$(kd+1)(k-d)^{m-1} \leq \alpha(m) \leq f_{d-1}(X)$$

and $m \leq \lfloor \log_{k-d} f_{d-1}(X) \rfloor$.

□

3 The Lower Bound

The proof of Theorem B_d depends on certain finite quotients of affine buildings constructed by Lubotzky, Samuels and Vishne [8], based on the Cartwright-Steger group [4] (see also [11] for a similar construction, as well as [3, 5, 7] for related results). In Section 3.1 we recall the definition and some properties of affine buildings of type $\tilde{A}_{d-1}$. In Section 3.2 we describe the relevant finite quotients and show that they have a large acyclicity radius.

3.1 Affine Buildings of Type $\tilde{A}_{d-1}$

Let F be a local field with a valuation $\nu : F \to \mathbb{Z}$ and a uniformizer π. Let $\mathcal{O}$ denote the ring of integers of F and $\mathcal{O}/\pi\mathcal{O} = \mathbb{F}_q$ be the residue field. A *lattice* L in the vector space $V = F^d$ is a finitely generated $\mathcal{O}$-submodule of V such that L contains a basis of V. Two lattices L_1 and L_2 are *equivalent* if $L_1 = \lambda L_2$ for some $0 \neq \lambda \in F$. Let $[L]$ denote the equivalence class of a lattice L. Two distinct equivalence classes $[L_1]$ and $[L_2]$ are *adjacent* if there exist representatives $L_1' \in [L_1]$, $L_2' \in [L_2]$ such that $\pi L_1' \subset L_2' \subset L_1'$. The affine building of type $\tilde{A}_{d-1}$ associated with F is the simplicial complex $\mathcal{B} = \mathcal{B}_d(F)$ whose vertex set $\mathcal{B}^0$ is the set of equivalence classes of lattices in V, and whose simplices are the subsets $\{[L_0], \ldots, [L_k]\}$ such that all pairs $[L_i], [L_j]$ are adjacent. It can be shown that $\{[L_0], \ldots, [L_k]\}$ forms a simplex iff there exist representatives $L_i' \in [L_i]$ such that

$$\pi L_k' \subset L_0' \subset \cdots \subset L_k' \quad . \tag{5}$$

It is well known that $\mathcal{B}$ is a contractible $(d-1)$-dimensional simplicial complex and that the link of each vertex is isomorphic to the order complex $A_{d-1}(\mathbb{F}_q)$ of all non-trivial proper linear subspaces of $\mathbb{F}_q^d$ (see e.g. [10, 7]). This implies that $\delta(\mathcal{B}) = q + 1$.

The *type function* $\tau : \mathcal{B}^0 \to \mathbb{Z}_d$ is defined as follows. Let $\mathcal{O}^d$ be the standard lattice in V. For any lattice L, there exists $g \in \mathrm{GL}(V)$ such that $L = g\mathcal{O}^d$. Define $\tau([L]) = \nu(\det(g)) \pmod{d}$. Let $\mathrm{dist}([L], [L'])$ denote the graph distance between $[L], [L'] \in \mathcal{B}^0$ in the 1-skeleton of $\mathcal{B}$. Let $\mathrm{dist}_1([L], [L'])$ denote the minimal t for which there exist $[L] = [L_0], \ldots, [L_t] = [L']$ such that $[L_i]$ and $[L_{i+1}]$ are adjacent in $\mathcal{B}$ and $\tau([L_{i+1}]) - \tau([L_i]) = 1$ for all $0 \le i \le t-1$.

Claim 3.1. *For two lattices L_1, L_2*

$$\mathrm{dist}_1([L_1], [L_2]) \le (d-1)\mathrm{dist}([L_1], [L_2]) \quad . \tag{6}$$

Proof: This follows directly from (5). Alternatively, let $v_1, \ldots, v_d$ be a basis of V and let $a_1, \ldots, a_d$ be integers such that $L_1 = \oplus_{i=1}^d \mathcal{O} v_i$ and $L_2 = \oplus_{i=1}^d \pi^{a_i} \mathcal{O} v_i$. Then

$$\mathrm{dist}([L_1], [L_2]) = \max_i a_i - \min_i a_i \tag{7}$$

and

$$\mathrm{dist}_1([L_1],[L_2]) = \sum_{i=1}^{d} a_i - d\min_i a_i \quad . \tag{8}$$

Now (6) follows from (7) and (8).

□

3.2 Finite Quotients of Affine Buildings

Let q be a prime power and let F be the local field $\mathbb{F}_q((y))$ with local ring $\mathcal{O} = \mathbb{F}_q[[y]]$. The construction of finite quotients of $\mathcal{B} = \mathcal{B}_d(F)$ in [8], depends on the remarkable Cartwright-Steger group $\Gamma < \mathrm{PGL}_d(F)$ (see [4]). We briefly recall the construction of Γ and some of its properties.

Let $\phi : \mathbb{F}_{q^d} \to \mathbb{F}_{q^d}$ denote the Frobenius automorphism. Extend ϕ to $\mathbb{F}_{q^d}(y)$ by defining $\phi(y) = y$. Then ϕ is a generator of the cyclic Galois group $\mathrm{Gal}(\mathbb{F}_{q^d}(y)/\mathbb{F}_q(y))$. Let $\mathcal{D}$ be the d^2-dimensional $\mathbb{F}_q(y)$-algebra given by $\mathcal{D} = \mathbb{F}_{q^d}[\sigma]$ with the relations $\sigma a = \phi(a)\sigma$ for all $a \in \mathbb{F}_{q^d}(y)$, and $\sigma^d = 1+y$. $\mathcal{D}$ is a division algebra that splits over the extension field $F = \mathbb{F}_q((y))$. Denote $\mathcal{D}(F) = \mathcal{D} \otimes F$, then there is an isomorphism $\mathcal{D}(F) \cong M_d(F)$ which in turn induces an isomorphism

$$\mathcal{D}(F)^\times / Z(\mathcal{D}(F)^\times) \cong \mathrm{PGL}_d(F) \quad . \tag{9}$$

Let $b_1 = 1 - \sigma^{-1} \in \mathcal{D}^\times$, and for $u \in \mathbb{F}_{q^d}^*$ let $b_u = u^{-1} b_1 u$. Let $g_u \in \mathcal{D}(F)^\times / Z(\mathcal{D}(F)^\times)$ denote the image of b_u under the quotient map. The Cartwright-Steger group Γ is the subgroup of $\mathcal{D}(F)^\times / Z(\mathcal{D}(F)^\times)$ generated by $\{g_u : u \in \mathbb{F}_{q^d}^*\}$. Utilizing the isomorphism (9), we also regard Γ as a subgroup of $\mathrm{PGL}_d(F)$. We shall use the following properties of Γ.

Theorem 3.2. *(Cartwright and Steger [4])*

a) Γ *acts simply transitively on the vertices of* $\mathcal{B}$.

b) Let $L_0 = \mathcal{O}^d$. *Then for* $g \in \Gamma$

$$\mathrm{dist}_1(g[L_0],[L_0]) = \min\{t : g = g_{u_1} \cdots g_{u_t} \text{ for some } u_1, \ldots, u_t \in \mathbb{F}_{q^d}^*\}.$$

□

The action of $\mathcal{D}$ upon itself by conjugation gives rise to a representation

$$\rho : \mathcal{D}(F)^{\times} \to \mathrm{GL}_{d^2}(F)$$

which factors through $\mathcal{D}(F)^{\times}/Z(\mathcal{D}(F)^{\times})$. Let $\xi_0, \ldots, \xi_{d-1}$ be a normal basis of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$, then $\{\xi_i \sigma^j\}_{i,j=0}^{d-1}$ is a basis of $\mathcal{D}(F)$ over F. An explicit computation (see Eq. (9) on page 975 in [8]) shows that with respect to this basis, $\rho(b_u)$ is a $d^2 \times d^2$ matrix whose entries are linear polynomials in $\frac{1}{y}$ over $\mathbb{F}_q$. Let $h(\lambda) \in \mathbb{F}_q[\lambda]$ be an irreducible polynomial which is prime to $\lambda(1+\lambda)$, and let $f = h(\frac{1}{y}) \in R_0 = \mathbb{F}_q[\frac{1}{y}]$ and $I = fR_0$. Write $\mathbf{1}_{d^2}$ for the $d^2 \times d^2$ identity matrix. Let

$$\Gamma(I) = \{ \ \gamma \in \Gamma : \rho(\gamma) \equiv \mathbf{1}_{d^2} (\mathrm{mod}\ f) \ \} \ .$$

This subgroup coincides with the congruence subgroup $\Gamma(I)$ as defined in Eq. (15) on p.979 in [8]. In particular $\Gamma/\Gamma(I)$ is isomorphic to a subgroup of $\mathrm{PGL}_d(R_0/fR_0)$. Let $\mathcal{B}_I = \Gamma(I) \backslash \mathcal{B}$ denote the resulting quotient complex. The vertex set $\mathcal{B}_I^0$ of $\mathcal{B}_I$ is the set of orbits of $\mathcal{B}^0$ under $\Gamma(I)$, i.e.

$$\mathcal{B}_I^0 = \{ \ \Gamma(I)[L] \ : \ [L] \in \mathcal{B}^0 \ \} \ .$$

A subset $\{\Gamma(I)[L_0], \ldots, \Gamma(I)[L_k]\}$ forms a simplex in $\mathcal{B}_I$ iff there exist $g_0, \ldots, g_k \in \Gamma(I)$ such that $\{g_0[L_0], \ldots, g_k[L_k]\}$ is a simplex in $\mathcal{B}$.
Note that

$$|\mathcal{B}_I^0| = (\Gamma : \Gamma(I)) \leq |\mathrm{PGL}_d(R_0/fR_0)| \ .$$

Let L be a lattice, and let

$$\ell_I = \min\{ \ \mathrm{dist}([L], g[L]) \ : \ 1 \neq g \in \Gamma(I) \ \} \ .$$

Clearly ℓ_I is independent of L since Γ is transitive and $\Gamma(I) \lhd \Gamma$.

Proposition 3.3.

$$\ell_I \geq \frac{\log_q |\mathcal{B}_I^0|}{(d-1)(d^2-1)} \ . \tag{10}$$

Proof: Let $t = \mathrm{dist}_1(g[L_0], [L_0])$. By Theorem 3.2b) there exist $u_1, \ldots, u_t \in \mathbb{F}_{q^d}^*$ such that $g = g_{u_1} \cdots g_{u_t}$. Let $C = (c_{ij}) = \rho(b_{u_1}) \cdots \rho(b_{u_t})$. The c_{ij}'s are polynomials in $\mathbb{F}_q[\frac{1}{y}]$ of degree at most t in $\frac{1}{y}$. By assumption $g \in \Gamma(I)$, hence $C = \mathbf{1}_{d^2} + fE$ for some $E \in M_{d^2}(R_0)$. If $c_{ij} \neq 0$ for some $i \neq j$, then $t \geq \deg_{1/y}(c_{ij}) \geq \deg_{1/y}(f)$. Otherwise C is a diagonal matrix. If it is a scalar matrix, then it must be the identity as Γ, being a lattice in $\mathrm{PGL}_d(F)$, has trivial center. Thus we can assume C is diagonal and non-scalar. Choose i, j such that $c_{ii} \neq c_{jj}$, then $t \geq \deg_{1/y}(c_{ii} - c_{jj}) \geq \deg_{1/y}(f)$. Thus, by (6)

$$\mathrm{dist}([L_0], g[L_0]) \geq \frac{1}{d-1}\mathrm{dist}_1(g[L], [L]) \geq$$

$$\frac{\deg_{1/y}(f)}{(d-1)} \geq \frac{\log_q |\mathrm{PGL}_d(R_0/fR_0)|}{(d-1)(d^2-1)} \geq$$

$$\frac{\log_q |\mathcal{B}_I^0|}{(d-1)(d^2-1)} \quad .$$

□

Proof of Theorem $\mathbf{B}_{d-1}$: Choose a sequence of irreducible polynomials $h_i(\lambda) \in \mathbb{F}_q[\lambda]$ such that $(h_i, \lambda(1+\lambda) = 1$ and $\deg h_i \to \infty$. Let $I_i = h_i(\frac{1}{y})R_0$ and let $X_i = \mathcal{B}_{I_i}$. The quotient map $\mathcal{B} \to X_i$ is clearly an isomorphism on balls of radius at most $\frac{\ell_{I_i}}{2} - 1$ in $\mathcal{B}$. Since $\mathcal{B}$ is contractible, it follows from Proposition 3.3 that for any vertex $v \in X_i^0$

$$r_v(X_i) \geq \frac{\ell_{I_i}}{2} - 1 \geq$$

$$\frac{\log_q |X_i^0|}{2(d-1)(d^2-1)} - 1 \quad .$$

We complete the proof by noting that if i is sufficiently large then $\ell_{I_i} \geq 4$, hence X_i is $(d-1)$-dimensional and $\delta(X_i) = \delta(\mathcal{B}) = q+1$.

□

4 Concluding Remarks

We proved a higher dimensional extension of the Moore bound, and showed that the Ramanujan Complexes constructed in [8] imply that this bound is tight up to a multiplicative factor. We mention several problems that arise from these results.

1. In Section 3.2 it is shown that for appropriately chosen ideals $I_i \triangleleft \mathbb{F}_q[\frac{1}{y}]$, the $(d-1)$-dimensional quotient complexes $X_i = \mathcal{B}_{I_i}$ satisfy

$$r_v(X_i) \geq C(d-1) \log_q |X_i^0| - 1$$

with $C(d-1) = \frac{1}{2(d-1)(d^2-1)}$. It seems likely that a more careful choice of the I_i's will lead to an improved bound on the constant. (Recall that in the 1-dimensional case, Ramanujan graphs [6] give the constant $\frac{2}{3}$, while $C(1) = \frac{1}{6}$).

2. While the construction of Ramanujan Graphs and the proof of Theorem B_1 depend on number theoretic tools, there is an elementary (but non-constructive) argument due to Erdős and Sachs (see e.g. Theorem III.1.4 in [1]) that shows the existence of a sequence of k-regular graphs $G_i = (V_i, E_i)$ with $|V_i| \to \infty$ such that $r(G_i) \geq \frac{1}{2} \log_{k-1} |V_i| - O(1)$. It would be interesting to obtain a similar result in the higher dimensional setting.

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